REPRESENTATION AND CHARACTER THEORY IN 2-CATEGORIES

NORA GANTER AND MIKHAIL KAPRANOV

Abstract. We develop a (2-)categorical generalization of the theory of group representations and characters. We categorify the concept of the trace of a linear transformation, associating to any endofunctor of any small category a set called its categorical trace. In a linear situation, the categorical trace is a vector space and we associate to any two commuting self-equivalences a number called their joint trace. For a group acting on a linear category $\mathcal{V}$ we define an analog of the character which is the function on commuting pairs of group elements given by the joint traces of the corresponding functors. We call this function the 2-character of $\mathcal{V}$. Such functions of commuting pairs (and more generally, n-tuples) appear in the work of Hopkins, Kuhn and Ravenel [HKR00] on equivariant Morava E-theories. We define the concept of induced categorical representation and show that the corresponding 2-character is given by the same formula as was obtained in [HKR00] for the transfer map in the second equivariant Morava E-theory.

1. Introduction

The goal of this paper is to develop a (2-)categorical generalization of the theory of group representations and characters. It is classical that a representation ϱ of a group G is often determined by its character

$$\chi(g) = tr(\varrho(g)),$$

which is a class function on G.

Remarkably, generalizations of character theory turn up naturally in the context of homotopy theory. Since this so called *Hopkins-Kuhn-Ravenel character theory* motivated much of our work, we will start with a short and very informal summary of it and some other homotopy theoretic topics. Fix a prime p, let n be a natural number, and let BG denote the classifying space of G. Assume that G is finite. In [HKR00],

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Hopkins, Kuhn and Ravenel computed the ring $E_n^*(BG)$, where E_n is a generalized cohomology theory (depending on p), which was introduced by Morava [Re98]. The first Morava E-theory is p-completed K-theory,

$$E_1 = K_{\hat{p}}.$$

Hopkins, Kuhn and Ravenel found that elements $\chi \in E_n^*(BG)$ are most naturally described as n-class functions, i.e. functions

$$\chi(g_1, \ldots, g_n)$$

defined on n-tuples of commuting elements of G and invariant under simultaneous conjugation. In the context of [HKR00], all the g_i are required to have p-power order. Hopkins, Kuhn and Ravenel actually make a much stronger case that the n-class functions occurring in this way should be viewed as generalized group characters: if $\alpha\colon H \hookrightarrow G$ is the inclusion of a subgroup, then there is a map

$$B\alpha\colon BH \to BG,$$

and in the stable homotopy category, one has a transfer map $\tau\alpha$ in the other direction. These maps make the correspondence $G \mapsto E_n(BG)$ into a Mackey functor. The map $E_n^*(\tau\alpha)$ sends a generalized character of H to a generalized character of G. If we stick with the analogy to classical character theory, it plays the role of the induced representation. Hopkins, Kuhn and Ravenel compute its effect on generalized characters. They find that it is described by the formula

$$E_n^*(\tau\alpha)(\chi)(g_1, \ldots, g_n) = \frac{1}{|H|} \sum_{\substack{s \in G \mid \\ s^{-1}\underline{g}s \in H^n}} \chi(s^{-1}g_1 s, \ldots, s^{-1}g_n s), \tag{1}$$

where $\underline{g} = (g_1, \ldots, g_n)$ is an n-tuple of commuting elements in G. For $n = 1$, this is the formula for the character of the induced representation, cf. [Ser77].

The number n is often referred to as the *chromatic level* of the theory, see [Rav92] for general background on the chromatic picture in homotopy theory. In the case $n = 2$, the theory E_2 is an example of an elliptic cohomology theory. For background on elliptic cohomology, we refer the reader to the introduction of [AHS01]).

Just as the representation ring $R(G)$ may be viewed as equivariant K-theory of the one point space, the ring $E_2(BG)$ is Borel equivariant E_2-theory of the one point space. Elliptic cohomology is a field at the intersection of several areas of mathematics, and there is a variety of fields that have motivated definitions of equivariant elliptic cohomology. To name a few, we have Devoto's definition, motivated by orbifold

string theory [Dev96], we have Grojnowski's work [Gro94], motivated by the theory of loop groups, we have the axiomatic approach in [GKV], involving principal bundles over elliptic curves, we have a connection to generalized Moonshine (cf. [G07], [Dev96], and [BT]), and we have recent constructions of Lurie and Gepner [Lu], [Gep], which satisfy axioms similar to those of [GKV] but formulated in the context of derived algebraic geometry. Lurie's construction naturally involves 2-groups. It is remarkable that each of these constructions, in one way or another, leads to class-functions on pairs of commuting elements of the group.

What is lacking in the above approaches is an analog of the notion of representation which would produce the generalized characters by means of some kind of trace construction. In this paper, we supply such a notion (for $n = 2$). It turns out that the right object to consider is an action of G on a category instead of a vector space or, more generally, on an object of a 2-category.

Our main construction is the so-called *categorical trace* of a functor $A : \mathcal{V} \to \mathcal{V}$ from a small category $\mathcal{V}$ to itself (or, more generally, of a 1-endomorphism of an object of a 2-category). This categorical trace is a set, denoted $\mathbb{T}r(A)$, see Definition 3.1. If k is a field, and the category $\mathcal{V}$ is k-linear, then $\mathbb{T}r(A)$ is a k-vector space. In the latter case, given two commuting self-equivalences $A, B : \mathcal{V} \to \mathcal{V}$, we define their *joint trace* $\tau(A, B)$ to be the ordinary trace of the linear transformation induced by B on the vector space $\mathbb{T}r(A)$. Here we assume that $\mathbb{T}r(A)$ is finite-dimensional.

For a group G acting on $\mathcal{V}$ this gives a 2-class function called the 2-character of the categorical representation $\mathcal{V}$. Among other things, we define an induction procedure for categorical representations and show that it produces the map (1) on the level of characters.

This very simple and natural construction ties in with other geometric approaches to elliptic cohomology. Already Segal, in his Bourbaki talk [Seg88], proposed to look for some kind of "elliptic objects" which are related to vector bundles in the same way as 2-categories are related to ordinary categories. While vector bundles can be equipped with connections and thus with the concept of parallel transport along paths, one expects elliptic objects to allow parallel transport along 2-dimensional membranes. Similarly, more recent works [TS04], [BDR04], [HK04] aiming at geometric definitions of elliptic cohomology, all involve 2-categorical constructions.

The present paper is kept at an elementary level and does not require any knowledge of homotopy theory (except for the final section 8 devoted to the comparison with [HKR00]). Nevertheless, the connection with homotopy theory and, in particular, with equivariant elliptic cohomology was important for us. It provided us with a motivation as well as with a possible future field of applications.

We also do not attempt to discuss actions of groups on n-categories for $n > 1$ which seem to be the right way to get $(n+1)$-class functions.

We have recently learned of a work in progress by Bruce Bartlett and Simon Willerton [BW] where, interestingly, the concept of the categorical trace also appears although the motivation is different.

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2. Background on 2-categories

2.1. Recall [ML98] that a 2-category $\mathcal{C}$ consists of the following data:

(1) A class of objects $\mathcal{O}b\,\mathcal{C}$.
(2) For any $x, y \in \mathcal{O}b\,\mathcal{C}$ a category $Hom_{\mathcal{C}}(x, y)$. Its objects are called 1-morphisms from x to y (notation $A : x \to y$). We will also use the notation $1Hom_{\mathcal{C}}(x, y)$ for $\mathcal{O}b\,Hom_{\mathcal{C}}(x, y)$. For any $A, B \in 1Hom_{\mathcal{C}}(x, y)$ morphisms from A to B in $Hom_{\mathcal{C}}(x, y)$ are called 2-morphisms from A to B (notation $\phi : A \Rightarrow B$). We denote the set of such morphisms by

$$2Hom_{\mathcal{C}}(A, B) = Hom_{Hom_{\mathcal{C}}(x,y)}(A, B).$$

The composition in the category $Hom_{\mathcal{C}}(x, y)$ will be denoted by $\circ_1$ and called the vertical composition. Thus if $\phi : A \Rightarrow B$ and $\psi : B \Rightarrow C$, then $\psi \circ_1 \phi : A \Rightarrow C$.
(3) The composition bifunctor

$$\begin{array}{rcl} Hom_{\mathcal{C}}(y, z) \times Hom_{\mathcal{C}}(x, y) & \to & Hom_{\mathcal{C}}(x, z) \\ (A, B) & \mapsto & A \circ_0 B. \end{array}$$

In particular, for a pair of 1-morphisms $A, B : x \to y$, a 2-morphisms $\phi : A \Rightarrow B$ between them, and a pair of 1-morphisms

$C, D : y \to z$ and a 2-morphism $\psi : C \Rightarrow D$ between them, there is a composition $\psi \circ_0 \phi : C \circ_0 A \Rightarrow D \circ_0 B$.

(4) The natural associativity isomorphism

$$\alpha_{A,B,C} : (A \circ_0 B) \circ_0 C \Rightarrow A \circ_0 (B \circ_0 C).$$

It is given for any three composable 1-morphisms A, B, C and satisfies the pentagonal axiom, see [ML98].

(5) For any $x \in \mathcal{O}b\,\mathcal{C}$, a 1-morphism $1_x \in 1Hom_{\mathcal{C}}(x, x)$ called the unit morphism, which comes equipped with 2-isomorphisms

$$\begin{aligned} \epsilon_\phi &: 1_x \circ_0 A \Rightarrow A, \quad \text{for any } A : y \to x, \\ \zeta_\psi &: B \circ 1_x \Rightarrow B, \quad \text{for any } B : x \to z, \end{aligned}$$

satisfying the axioms of [ML98].

We denote by $1\mathrm{Mor}\mathcal{C}$ and $2\mathrm{Mor}\mathcal{C}$ the classes of all 1- and 2-morphisms of $\mathcal{C}$. We say that $\mathcal{C}$ is strict if all the $\alpha_{A,B,C}$, ϵ_ϕ and ζ_ψ are identities (in particular the source and target of each of them are equal). It is a theorem of Mac Lane and Paré that every 2-category can be replaced by a (2-equivalent) strict one. See [ML98] for details.

2.2. **Examples.** (a) The 2-category $\mathcal{C}at$ has, as objects, all small categories, as morphisms their functors and as 2-morphisms natural transformations of functors. This 2-category is strict. We will use the notation $Fun(\mathcal{A}, \mathcal{B})$ for the set of functors between categories $\mathcal{A}$ and $\mathcal{B}$ (i.e., 1-morphisms in $\mathcal{C}at$) and $NT(F, \Phi)$ for the set of natural transformations between functors F and Φ. Most of the examples of 2-categories can be embedded into $\mathcal{C}at$: a 2-category $\mathcal{C}$ can be realized as consisting of categories with some extra structure.

(b) Let k be a field. The 2-category $2Vect_k$, see [KV94] has, as objects, symbols $[n]$, $n = 0, 1, 2, ...$ For any two such objects $[m], [n]$ the category $Hom([m], [n])$ has, as objects, 2-matrices of size m by n, i.e., matrices of vector spaces $A = \|A_{ij}\|$, $i = 1, ..., m$, $j = 1, ..., n$. Morphisms between 2-matrices A and B of the same size are collections of linear maps $\phi = \{\phi_{ij} : A_{ij} \to B_{ij}\}$. Composition of 2-matrices is given by the formula

$$(A \circ B)_{ij} = \bigoplus_l A_{il} \otimes B_{lj}.$$

This 2-category is not strict. An explicit strict replacement was constructed in [Elga].

(c) The 2-category $\mathcal{B}im$ has, as objects, associative rings. If R, S are two such rings, then $Hom_{\mathcal{B}im}(R, S)$ is the category of (R, S)-bimodules.

The composition bifunctor

$$Hom(S,T) \times Hom(R,S) \to Hom(R,T)$$

is given by the tensor product:

$$(M,N) \mapsto N \otimes_S M.$$

This 2-category is also not strict.

Relation to $\mathcal{C}at$: To a ring R we associate the category Mod-R of right R-modules. Then each (R,S)-bimodule M defines a functor

$$\text{Mod-}R \to \text{Mod-}S, \quad P \mapsto P \otimes_R M.$$

The 2-category $2Vect_k$ is realized inside $\mathcal{B}im$ by associating to $[m]$ the ring $k^{\oplus m}$. An m by n 2-matrix is the same as a $(k^{\oplus m}, k^{\oplus n})$-bimodule.

We will denote by $\mathcal{B}im_k$ the sub-2-category in $\mathcal{B}im$ formed by k-algebras as objects and the same 1- and 2-morphisms as in $\mathcal{B}im$.

(d) Let X be a CW-complex. The Poincare 2-category $\Pi(X)$ has, as objects, points of X, as 1-morphisms Moore paths $[0,t] \to X$ and as 2-morphisms homotopy classes of homotopies between Moore paths.

We will occasionally use the concept of a (strong) 2-functor $\Phi\colon \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$. Such a 2-functor consists of maps $\mathrm{Ob}\mathcal{C} \to \mathrm{Ob}\mathcal{D}$, $1\mathrm{Mor}\mathcal{C} \to 1\mathrm{Mor}\mathcal{D}$ and $2\mathrm{Mor}\mathcal{C} \to 2\mathrm{Mor}\mathcal{D}$ preserving the composition of 2-morphisms and preserving the composition of 1-morphisms up to natural 2-isomorphisms. See [ML98] for details.

2.3. **2-categories with extra structure.** We recall the definition of an enriched category from [ML98] or [Kel82]. Let $(\mathcal{A}, \otimes, S)$ be a closed symmetric monoidal category, so $\otimes$ is the monoidal operation and S is a unit object.

Definition 2.1. A category enriched over $\mathcal{A}$ (or simply a $\mathcal{A}$-category) C is defined in the same way as a category, with the morphism sets replaced by objects $\hom(X,Y)$ of $\mathcal{A}$ and composition replaced by $\mathcal{A}$-morphisms

$$\hom(X,Y) \otimes \hom(Y,Z) \to \hom(X,Z),$$

with units

$$1_X\colon S \to \hom(X,X),$$

such that the usual associativity and unit diagrams commute.

Example 2.2. Categories enriched over the category of abelian groups are commonly known as pre-additive categories. By an additive category one means a pre-additive category possessing finite direct sums.

If k is a field, categories enriched over the category of k-vector spaces are known as k-linear categories.

Example 2.3. A strict 2-category is the same as a category enriched over the category of small categories with $\otimes$ being the direct product of categories (cf. [St87]).

Definition 2.4. Let $\mathcal{A}$ be a category. A *strict* 2-*category* $\mathcal{C}$ *enriched over* $\mathcal{A}$, or shorter an $\mathcal{A}$-2-*category*, is a category enriched over the category of small $\mathcal{A}$-categories.

Definition 2.5. We define a strict pre-additive 2-category to be a 2-category enriched over the category of abelian groups. Let k be a field. Then a (strict) k-linear 2-category is defined to be a 2-category enriched over the category of k-vector spaces. Weak additive and k-linear 2-categories are defined in a similar way.

We will freely use the concept of a triangulated category [Nee01], [GM03]. If $\mathcal{D}$ is triangulated, then we denote by $X[i]$ the i-fold iterated shift (suspension) of an object X in $\mathcal{D}$. We will denote

$$Hom^{\bullet}_{\mathcal{D}}(X,Y) = \bigoplus_i Hom_{\mathcal{D}}(X, Y[i]).$$

We will call a 2-category $\mathcal{C}$ triangular if each $Hom_{\mathcal{C}}(x,y)$ is made into a triangulated category and the composition functor is exact in each variable.

2.4. **Examples.** (a) The 2-category $\mathcal{B}im$ is additive. The 2-categories $2Vect_k$ and $\mathcal{B}im_k$ are k-linear.

(b) Define the 2-category $\mathcal{DB}im$ to have the same objects as $\mathcal{B}im$, i.e., associative rings. The category $Hom_{\mathcal{DB}im}(R,S)$ is defined to be the derived category of complexes of (R,S)-bimodules bounded above. The composition is given by the derived tensor product:

$$(M,N) \mapsto N \otimes^L_S M.$$

This gives a triangular 2-category.

(c) The 2-category $\mathcal{V}ar_k$ has as objects smooth projective algebraic varieties over k. If X, Y are two such varieties, then

$$Hom_{\mathcal{V}ar_k}(X,Y) = D^b\mathcal{C}oh(X \times Y)$$

is the bounded derived category of coherent sheaves on $X \times Y$. If $\mathcal{K} \in D^b\mathcal{C}oh(Y \times Z)$ and $\mathcal{L} \in D^b\mathcal{C}oh(X \times Y)$, then their composition is defined by the derived convolution

$$\mathcal{K} * \mathcal{L} = Rp_{13*}(p_{12}^* \mathcal{L} \otimes^L p_{23}^* \mathcal{K}),$$

where p_{12}, p_{13}, p_{23} are the projections of $X \times Y \times Z$ to the products of two factors. This again gives a triangular 2-category.

Relation to $\mathcal{C}at$: To every variety X we associate the category $D^b\mathcal{C}oh(X)$. Then every sheaf $\mathcal{K} \in D^b\mathcal{C}oh(X \times Y)$ ("kernel") defines a functor

$$F_{\mathcal{K}} : D^b\mathcal{C}oh(X) \to D^b\mathcal{C}oh(Y), \quad \mathcal{F} \mapsto Rp_{2*}(p_1^* \mathcal{F} \otimes^L \mathcal{K}),$$

and $F_{\mathcal{K} * \mathcal{L}}$ is naturally isomorphic to $F_{\mathcal{K}} \circ F_{\mathcal{L}}$. It is not known, however, whether the natural map

$$Hom_{D^b\mathcal{C}oh(X,Y)}(\mathcal{K}, \mathcal{L}) \to NT(F_{\mathcal{K}}, F_{\mathcal{L}})$$

is a bijection for arbitrary $\mathcal{K}, \mathcal{L}$. So in practice the source of this map is used as a substitute for its target.

(d) The 2-category $\mathbb{R}\mathcal{A}n_k$ has, as objects, real analytic manifolds. For any two such manifolds X, Y the category $Hom_{\mathcal{CW}}(X, Y)$ is defined to be $D^b\mathcal{C}onstr(X \times Y)$, the bounded derived category of ($\mathbb{R}$-) constructible sheaves of k-vector spaces on $X \times Y$, see [KS94], Sect. 8.4., for background on constructible sheaves. The composition is defined similarly to the above, with p_{ij}^* understood as sheaf-theoretic direct images rather than $\mathcal{O}$-module-theoretic direct images. This is a triangular 2-category.

Relation to $\mathcal{C}at$: To every real analytic manifold X we associate the category $D^b\mathcal{C}onstr(X)$. Then, as in (c), any sheaf

$$\mathcal{K} \in D^b\mathcal{C}onstr(X \times Y)$$

can be considered as a "kernel" defining a functor

$$D^b\mathcal{C}onstr(X) \to D^b\mathcal{C}onstr(Y).$$

(e) Let $\mathcal{A}b$ denote the category of all abelian categories. For any such categories $\mathcal{A}, \mathcal{B}$ the category $Fun(\mathcal{A}, \mathcal{B})$ is again abelian: a sequence of functors is exact if it takes any object into an exact sequence. So we have a triangular 2-category $\mathcal{D}\mathcal{A}b$ with same objects as $\mathcal{A}b$ but $Hom(\mathcal{A}, \mathcal{B}) = D^b Fun(\mathcal{A}, \mathcal{B})$.

3. The categorical trace

3.1. **The main definition.** As motivation consider the 2-category $2Vect_k$. In this situation there is a naïve way to define the "trace" of a 1-automorphism, namely as direct sum of the diagonal entries of

the matrix. This naïve notion of trace is equivalent to the following definition that makes sense in any 2-category $\mathcal{C}$:

Definition 3.1. Let $\mathcal{C}$ be a 2-category, x an object of $\mathcal{C}$ and $A : x \to x$ a 1-endomorphism of x. The categorical trace of A is defined as

$$\mathbb{T}r(A) = 2Hom_{\mathcal{C}}(1_x, A).$$

If $\mathcal{C}$ is triangular, we write

$$\mathbb{T}r^i(A) = \mathbb{T}r(A[i]), i \in \mathbb{Z}, \quad \mathbb{T}r^\bullet(A) = \bigoplus_i \mathbb{T}r^i(A).$$

Remark 3.2 (Functoriality)**.** Note that for each x, the categorical trace defines a functor

$$\begin{array}{rcl} \mathbb{T}r\colon 1End(x) & \to & Set \\ \phi \in 2Hom(A, B) & \mapsto & \phi_*, \end{array}$$

where

$$\phi_*\colon \mathbb{T}r(A) \to \mathbb{T}r(B)$$

is given by composition with ϕ. A priori, $\mathbb{T}r$ is set valued, but if we assume $\mathcal{C}$ to be enriched over a category $\mathcal{A}$ (cf. Definition 2.4), $\mathbb{T}r$ takes values in $\mathcal{A}$. We will often assume that $\mathcal{C}$ is k-linear for a fixed field k.

3.2. **Examples of the categorical trace.** Our first example is the motivational example mentioned above.

Example 3.3 (2-vector spaces)**.** Let $\mathcal{C} = 2\text{-Vect}_k$ and $x = [n]$. Then A is an $n \times n$ matrix $A = (A_{ij})$, where the A_{ij} are vector spaces. In this case,

$$\mathbb{T}r(A) = \bigoplus_{i=1}^{n} A_{ii}.$$

Example 3.4 (Categories)**.** Let $\mathcal{C} = \mathcal{C}at$ and $x = \mathcal{V}$ a category, so $A : \mathcal{V} \to \mathcal{V}$ is an endofunctor. Then $\mathbb{T}r(A) = NT(\mathrm{id}_{\mathcal{V}}, A)$ is the set of natural transformations from the identity functor to A.

Example 3.5 (Bimodules)**.** Let $\mathcal{C} = \mathcal{DB}im$, so that $x = R$ is a ring, and $A = M$ is an R-bimodule. Then

$$\mathbb{T}r^\bullet(A) = Ext^\bullet_{R \otimes R^{op}}(R, M)$$

is the Hochschild cohomology of R with coefficients in M, see [Lo98].

Example 3.6 (Varieties)**.** Let $\mathcal{C} = \mathcal{V}ar_k$, $x = X$ be a variety, and $A = \mathcal{K}$ be a complex of coherent sheaves on $X \times X$. Then

$$\mathbb{T}r^\bullet(A) = \mathbb{H}^\bullet(X, i^!(\mathcal{K})).$$

here $i : X \to X \times X$ is the diagonal embedding, $i^!$ is the right adjoint of i_*, and $\mathbb{H}$ is the hypercohomology. In particular, if $\mathcal{K}$ is a vector bundle on $X \times X$ situated in degree 0, then

$$\mathbb{T}r^\bullet(A) = H^\bullet(X, \mathcal{K}|_\Delta)$$

is the cohomology of the restriction of $\mathcal{K}$ to the diagonal.

3.3. **The center of an object.** The set $\mathbb{T}r(1_x)$ will be called the center of x and denoted $Z(x)$. It is closed under both compositions $\circ_0$ and $\circ_1$. The following fact is well known [ML98].

Proposition 3.7. *The operations $\circ_0$ and $\circ_1$ on $Z(x)$ coincide and make it into a commutative monoid.*

Thus, if $\mathcal{C}$ is pre-additive, then $Z(x)$ is a commutative ring and for each $A : x \to x$ the group $\mathbb{T}r(A)$ is a $Z(x)$-module.

3.4. **Examples.** (a) If $\mathcal{C} = \mathcal{A}b$ and $x = \mathcal{V}$ is an abelian category, then $Z(\mathcal{V})$, i.e., the ring of natural transformations from the identity functor to itself is known as the Bernstein center of $\mathcal{V}$, see [Ber84]

(b) If $\mathcal{C} = \Pi(X)$ is the Poincare 2-category of a CW-complex X, then $Z(x) = \pi_2(X, x)$ is the second homotopy group. Proposition 3.7 is the categorical analog of the commutativity of π_2.

3.5. **Conjugation invariance of the categorical trace.** In this section we assume for simplicity that the 2-category $\mathcal{C}$ is strict. Recall [ML98] that a 1-morphism $B : y \to x$ is called an equivalence if there exist a 1-morphism $C : x \to y$ called *quasi-inverse* and 2-isomorphisms $u : 1_x \Rightarrow BC$, $v : 1_y \Rightarrow CB$. For any object x, the 1-morphism $B = 1_x$ is an equivalence with $C = 1_x$ and u, v the isomorphisms from 2.1 (4). If B and B' are composable 1-morphisms, which are equivalences with quasi-inverses (C, u, v) and (C', u', v') respectively, then $B'' \circ_0 B$ is an equivalence with quasi-inverse

$$(C \circ_0 C', (B' \circ_0 u \circ_0 C') \circ_1 u', (C \circ_0 v' \circ_0 B) \circ_1 v)\,. \tag{2}$$

Proposition 3.8 (Conjugation invariance)**.** *(a) Let $A : x \to x$ be a 1-endomorphism and $B : x \to y$ an equivalence with quasi-inverse C. Then the rule*

$$(\phi : 1_x \Rightarrow A) \mapsto (B \circ_0 \phi \circ_0 C) \circ_1 u$$

defines a bijection of sets

$$\psi(B,C,u,v)\colon \mathbb{T}r(A) \to \mathbb{T}r(BAC).$$

By abuse of notation, we will write $\psi(B)$ when C, u and v are clear from the context.
(b) Assume that B and B' are 1-endomorphisms of x and that both of them are equivalences. Then we have

$$\psi(B' \circ_0 B) = \psi(B') \circ \psi(B).$$

(c) We have $\psi(1_x) = \mathrm{id}$.

Proof. (a) To explain the formula, note that we can view u as a 2-morphism $1_y \Rightarrow BC = B \circ 1_x \circ C$, while

$$B \circ_0 \phi \circ_0 C : B \circ 1_x \circ C \Rightarrow B \circ A \circ C.$$

Since u is a 2-isomorphism, composing with u is a bijection.

(b) This follows from the definition of $\psi(B)$ together with (2).

(c) Obvious. □

Proposition 3.9. *Let $\mathcal{C}$ be an additive 2-category and $A, A' : x \to x$ be two 1-morphisms. Then*

$$\mathbb{T}r(A \oplus A') = \mathbb{T}r(A) \oplus \mathbb{T}r(A').$$

This is an immediate consequence of the fact that $\mathcal{H} = Hom_{\mathcal{C}}(x,x)$ is an additive category, and that therefore

$$Hom_{\mathcal{H}}(1_x, A \oplus A') = Hom_{\mathcal{H}}(1_x, A) \oplus Hom_{\mathcal{H}}(1_x, A').$$

3.6. **The joint trace.** In the situation of Proposition 3.5 assume that A and B *commute*, i.e., that we are given a 2-isomorphism

$$\eta : B \circ A \Rightarrow A \circ B.$$

Then we have a map

$$B_* : \mathbb{T}r(A) \to \mathbb{T}r(A),$$

defined as the composition

$$\mathbb{T}r(A) \xrightarrow{\psi(B)} \mathbb{T}r(BAC) \xrightarrow{\mathbb{T}r(\eta \circ_0 1)} \mathbb{T}r(ABC) \xrightarrow{\mathbb{T}r(1 \circ_0 u^{-1})} \mathbb{T}r(A).$$

Assume now that the 2-category $\mathcal{C}$ is k-linear for a field k. Then $\mathbb{T}r(A)$ is a k-vector space, and B_* is a linear operator. Let us further assume that $\mathbb{T}r(A)$ is finite-dimensional. Then we define the *joint trace* of A and B to be the following element of k:

$$\tau(A,B) = \mathrm{Trace}\{B_* : \mathbb{T}r(A) \to \mathbb{T}r(A)\}.$$

It depends on the choice of the commutativity isomorphism η, as well as on the equivalence data for Φ.

4. 2-representations and their characters

4.1. **2-representations.** Let G be a group. We view G as a 2-category with one object, pt, the set of 1-morphisms $Hom(\mathrm{pt}, \mathrm{pt}) = G$ and all the 2-morphisms being the identities of the above 1-morphisms.

Definition 4.1. Let $\mathcal{C}$ be a 2-category. A 2-representation of G in $\mathcal{C}$ is a strong 2-functor from G to $\mathcal{C}$. More explicitly, this is a system of the following data:

(a) an object V of $\mathcal{C}$,
(b) for each element $g \in G$, a 1-automorphism $\rho(g)$ of V,
(c) for any pair of elements (g, h) of G a 2-isomorphism

$$\phi_{g,h} \colon (\rho(g) \circ \rho(h)) \overset{\cong}{\Longrightarrow} \rho(gh),$$

(d) and a 2-isomorphism

$$\phi_1 \colon \rho(1) \overset{\cong}{\Longrightarrow} \mathrm{id}_c,$$

such that the following conditions hold

(e) for any $g, h, k \in G$ we have

$$\phi_{(gh,k)}(\phi_{g,h} \circ \rho(k)) = \phi_{(g,hk)}(\rho(g) \circ \phi_{h,k})$$

(associativity); we also write $\phi_{g,h,k}$,
(f) we have

$$\phi_{1,g} = \phi_1 \circ \rho(g) \quad \text{and} \quad \phi_{g,1} = \rho(g) \circ \phi_1.$$

Note that this definition is the special case of the concept of a representation of a 2-group as defined by Elgueta [Elgb, Def.4.1]. This case corresponds to the 2-group being discrete, i.e., being reduced to an ordinary group. Compare also [Del97, §0]. If $\mathcal{D}$ and $\mathcal{C}$ are 2-categories, then strong 2-functors from $\mathcal{D}$ to $\mathcal{C}$ form a 2-category $\mathfrak{Hom}(\mathcal{D}, \mathcal{C})$, see [Ha72, Def. I.1.9] for strict 2-categories or [Ben68] for the general case. In particular, 2-representations of G in $\mathcal{C}$ form a 2-category. We will denote it by $2Rep_{\mathcal{C}}(G)$. We understand that the implications of this fact will be spelled out in detail in [BW].

4.2. **The category of equivariant objects.** Consider the particular case of Definition 4.1 when $\mathcal{C} = \mathcal{C}at$ is the 2-category of (small) categories. Then a 2-representation of G in $\mathcal{C}$ is the same as an action of G on a category $\mathcal{V}$. In other words, each

$$\rho(g) : \mathcal{V} \to \mathcal{V}$$

is a functor and each $\phi_{g,h}$ is a natural transformation. We will call a category with a G-action a *categorical representation* of G and will denote by $2Rep(G) = 2Rep_{\mathcal{C}at}(G)$ the 2-category formed by categorical representations.

In this section, we formulate the categorical analogue the concept of the subspace of G-invariants of a representation.

Definition 4.2. Fix a category $\underline{1}$ with one object and one morphism. The *trivial* 2*-representation of* G is given by the unique action of G on $\underline{1}$. We will also denote it by $\underline{1}$. Let ρ be an action of G on $\mathcal{V}$. We define the *category of* G*-equivariant objects in* $\mathcal{V}$, denoted $\mathcal{V}^G$, to be the category of G-functors from $\underline{1}$ to ρ:

$$\mathcal{V}^G = Hom_{2Rep(G)}(\underline{1}, \rho).$$

This definition spells out to the following. An object of $\mathcal{V}^G$ consists of an object $X \in \mathrm{Ob}(\mathcal{V})$ and a system

$$(\epsilon_g \colon X \to \rho(g)(X),\ g \in G)\,,$$

where ϵ_g are isomorphisms satisfying the following compatibility conditions: First, it is required that for $g = 1$ we have

$$\epsilon_1 = \phi_{1,X}^{-1} : X \mapsto \rho(1)(X).$$

Second, it is required that for any $g, h \in G$ the diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon_g"] \arrow[d, "\epsilon_{gh}"'] & \rho(g)(X) \arrow[d, "\rho(g)(\epsilon_h)"] \\ \rho(gh)(X) & \rho(g)(\rho(h)(X)) \arrow[l, "\phi_{g,h,X}"] \end{tikzcd}$$

is commutative.

Example 4.3. Let $\mathcal{W}$ be a category. Define the trivial action of G on $\mathcal{W}$ by taking all $\rho(g)$ and $\phi_{g,h}$ to be the identities. Then a G-equivariant object in $\mathcal{W}$ is the same as a representation of G in $\mathcal{W}$, i.e., an object $X \in \mathcal{W}$ and a homomorphism $G \to \mathrm{Aut}_{\mathcal{W}}(X)$.

Proposition 4.4. *Let* $\mathcal{W}$ *be a category equipped with trivial* G*-action as in Example 4.3. Then we have an equivalence of categories*

$$Hom_{2Rep(G)}(\mathcal{W}, \mathcal{V}) \simeq Hom_{\mathcal{C}at}(\mathcal{W}, \mathcal{V}^G).$$

In particular, taking $\mathcal{W} = \mathrm{pt}$ *(the category with one object and one morphism), we get*

$$\mathcal{V}^G \simeq Hom_{2Rep(G)}(\mathrm{pt}, \mathcal{V}).$$

In plain words, this means that any G-functor from $\mathcal{W}$ to $\mathcal{V}$ factors through the forgetful functor

$$i_{\mathcal{V}} : \mathcal{V}^G \to \mathcal{V}.$$

In 2-categorical terms, this can be formulated by saying that the 2-functor

$$I_G : 2Rep(G) \to \mathcal{C}at, \quad \mathcal{V} \mapsto \mathcal{V}^G,$$

is right 2-adjoint (in the sense of [Ha72, Def. I.1.10]), to the 2-functor $\mathcal{C}at \to 2Rep(G)$ associating to any $\mathcal{W}$ the same category $\mathcal{W}$ with trivial G-action.

Proof : This follows at once from the definition of the Hom-categories in $2Rep(G)$, which are particular cases of Hom-categories in 2-categories of 2-functors, see *loc. cit.* Def. I.1.9. Indeed, denote by $\widetilde{\rho}$ the trivial action of G on $\mathcal{W}$. Then a G-functor $F : \mathcal{W} \to \mathcal{V}$ gives, for each object $X \in \mathcal{W}$, an object $F(X) \in \mathcal{W}$ together with isomorphisms

$$u_{g,X} : F(\widetilde{\rho}(g)(X)) \to \rho(g)(F(X)),$$

satisfying the compatibility condition for each pair $g, h \in G$. Since $\widetilde{\rho}(g)(X) = X$, the system formed by $F(X)$ and the $u_{g,X}$ gives an equivariant object of $\mathcal{V}$. We leave further details to the reader. $\square$

Remark 4.5. The concept of the category of equivariant objects relates our approach to 2-representations with a different approach due to Ostrik [Ost01]. If k is a field, then finite-dimensional linear representations of G over k form a monoidal category $(\mathcal{R}ep(G), \otimes)$ with respect to the usual tensor product. In *loc. cit.* it was proposed to study module categories over $\mathcal{R}ep(G)$. In our situation, given a G-action on a k-linear additive category $\mathcal{V}$, the category $\mathcal{V}^G$ is naturally a module category over $\mathcal{R}ep(G)$. In other words, the tensor product of a G-representation and a G-equivariant object is again a G-equivariant object. It seems that in general, the passage from a G-category $\mathcal{V}$ to the $\mathcal{R}ep(G)$-module category $\mathcal{V}^G$ leads to some loss of information. However, in some particular cases, the two approaches are equivalent, see Remark 7.4 below.

4.3. **Characters of 2-representations.** We are now ready to define the categorical character of a 2-representation. To motivate the discussion of this section, we start with a reminder of classical character theory.

4.3.1. *Group characters and class functions.* We fix a field k of characteristic 0 containing all roots of unity. Let G be a group. Recall that a function $f\colon G \to k$ is called a *class function* if it is invariant under conjugation.

Notation 4.6. We denote by $\mathrm{Cl}(G;k)$ the ring of class functions on G. As before, let $\mathcal{R}ep(G)$ be the category of finite-dimensional representations of G over k. We write $R(G)$ for its Grothendieck ring $K(\mathcal{R}ep(G))$.

If $\rho : G \to Aut(V)$ is a representation, then its character

$$\begin{array}{rcl}\chi_V\colon G & \to & k \\ g & \mapsto & \mathrm{Trace}(\rho(g))\end{array}$$

is a class function. The following is well known [Ser77].

Proposition 4.7. *If G is finite, then the correspondence $V \mapsto \chi_V$ induces an isomorphism of rings*

$$R(G) \otimes k \to \mathrm{Cl}(G;k).$$

4.3.2. *The categorical character.* The classical definitions discussed in the previous section suggest the following analogues for 2-representations:

Definition 4.8. Let ρ be a 2-representation of G. We define the categorical character of ρ to be the assignment

$$g \mapsto \mathbb{T}r(\rho(g)).$$

We now discuss the sense in which the categorical character is a class function. First we recall the definition of the inertia groupoid of G:

Definition 4.9. Let G be a group. The *inertia groupoid* $\Lambda(G)$ of G is the category that has as objects the elements of G and

$$Hom_{\Lambda(G)}(u,v) = \left\{g \in G : \; v = gug^{-1}\right\}.$$

Proposition 4.10. *Let $\mathcal{C}$ be a 2-category and let ρ be a 2- representation of G in $\mathcal{C}$. Then the categorical character of ρ is a functor from the inertia groupoid $\Lambda(G)$ to the category of sets:*

$$\mathbb{T}r(\rho)\colon \Lambda(G) \to \mathrm{Set}.$$

If $\mathcal{C}$ is enriched over a category $\mathcal{A}$, then this functor takes values in $\mathcal{A}$. In other words, for any $f, g \in G$ there is an isomorphism

$$\psi(g) = \psi_f(g)\colon \mathbb{T}r(\rho(f)) \to \mathbb{T}r(\rho(gfg^{-1})),$$

and these isomorphisms satisfy

(a) $\psi(gh) = \psi(g) \circ \psi(h)$ *and*

(b) $\psi(1) = \mathrm{id}_{\rho(f)}$.

Proof. Pick $f, g \in G$ and write A for $\rho(f)$, B for $\rho(g)$, C for $\rho(g^{-1})$ and define

$$u\colon 1_c \to BC$$

as the composite of maps from Definition 4.1:

$$u := \phi_{g,g^{-1}}^{-1}\phi_1^{-1}.$$

With this notation, Proposition 3.8 (a) implies the existence of an isomorphism

$$\psi'\colon \mathbb{T}r(\rho(f)) \xrightarrow{\cong} \mathbb{T}r(\rho(g)\rho(f)\rho(g^{-1})).$$

Composed with $\mathbb{T}r(\phi_{g,f,g^{-1}})$ this gives the desired map $\psi(g)$. Properties (a) and (b) of $\psi(g)$ follow from Proposition 3.8 (b) and (c). $\square$

Remark 4.11. By regarding G as a discrete topological space, we can consider the correspondence $g \mapsto \mathbb{T}r(\rho(g))$ as a sheaf of sets on G. If G is a topological or algebraic group, there are natural situations when $\mathbb{T}r(\rho)$ is a sheaf on G in the corresponding stronger sense, equivariant under conjugation, see Subsection 5.3 below for an example.

Definition 4.12. If ρ is a 2-representation in a k-linear 2-category with finite-dimensional 2-Hom(ϕ, ψ), we define the categorical character of ρ to be the function χ_ρ on pairs of commuting elements given by the joint trace of $\rho(g)$ and $\rho(h)$:

$$\chi_\rho(g,h) = \tau(\rho(g), \rho(h)) = \mathrm{Trace}\big\{\psi(h) : \mathbb{T}r(\rho(g)) \to \mathbb{T}r(\rho(g))\big\}.$$

Note that

$$\chi_\rho(s^{-1}gs, s^{-1}hs) = \chi_\rho(g,h). \tag{3}$$

This can be formulated as follows.

Definition 4.13. Let G be a group and R be a commutative ring. A 2-class function on G with values in R is a function $\chi(g,h)$ defined on pairs of commuting elements of G and invariant under simultaneous conjugation, as in (3). The ring of such functions will be denoted $2\,\mathrm{Cl}(G;R)$.

Thus the categorical character is a 2-class function.

5. Examples

5.1. **1-dimensional 2-representations.** Let k be a field and

$$c : G \times G \to k^*$$

be a 2-cocycle, i.e., it satisfies the identity

$$c(g_1g_2, g_3)c(g_1, g_2) = c(g_1, g_2g_3)c(g_2, g_3).$$

We then have an action $\rho = \rho_c$ of G on $Vect_k$. By definition, for $g \in G$ the functor $\rho(g) : Vect_k \to Vect_k$ is the identity, while

$$\phi_{g,h} \colon \mathrm{id} = \rho(g) \circ \rho(h) \Rightarrow \rho(gh) = \mathrm{id}$$

is the multiplication with $c(g, h)$, and ϕ_1 is the multiplication by $c(1, 1)$. The cocycle condition for c is equivalent to Condition (e) of Definition 4.1, while Condition (f) follows because

$$c(1, 1g) \cdot c(1, g) = c(1, 1) \cdot c(1, g)$$

implies that

$$c(1, g) = c(1, 1)$$

and similarly

$$c(g, 1) = c(1, 1).$$

Cohomologous cocycles define equivalent 2-representations, and it is easy to see that $H^2(G, k^*)$ is identified with the set of G-actions on Vect_k modulo equivalence. Compare [Kap95].

We now find the categorical character and the 2-character of ρ_c. First of all, the functor $\rho_c(g)$ being the identity,

$$\mathbb{T}r(\rho_c(g)) = k.$$

Next, the equivariant structure on $\mathbb{T}r(\rho_c)$ was defined in the proofs of Propositions 3.8 (a) and 4.10 to be the composition

$$\mathbb{T}r(\rho_c(f)) \xrightarrow{\tilde{u}} \mathbb{T}r(\rho_c(g)\rho_c(f)\rho_c(g^{-1})) \longrightarrow \mathbb{T}r(\rho_c(gfg^{-1})).$$

Here $\tilde{u}$ is induced by the 1-composition with

$$u \colon 1_{\mathrm{Vect}_k} \Rightarrow BC,$$

where $B = \rho_c(g)$ and $C = \rho_c(g^{-1})$. In our case,

$$u = c(g, g^{-1})^{-1}$$

(multiplication with a scalar). The second map is induced by

$$\phi_{g,f,g^{-1}} = c(g, f)c(gf, g^{-1}),$$

see Definition 4.1 (e). As a result we have

Proposition 5.1. *For any two commuting elements $f, g \in G$, we have*

$$\chi_{\rho_c}(f, g) = c(g, f)c(gf, g^{-1})c(g, g^{-1})^{-1}.$$

Notice also the following fact which extends Example 4.3.

Proposition 5.2 (compare [Elgb])**.** *Let ρ_c be the one-dimensional* 2-*representation of G on $\mathcal{V} = \mathrm{Vect}_k$ corresponding to c. Then objects of $\mathcal{V}^G$ are the same as projective representations of G with central charge c, i.e., pairs $(V, \varphi : G \to \mathrm{Aut}(V))$, where V is a k-vector space, and φ is a map satisfying*

$$\varphi(gh) = \varphi(g)\varphi(h) \cdot c(g, h).$$

5.2. **Representations on 2-vector spaces.** 2-representations ρ_c from Section 5.1 can be viewed as acting on the 1-dimensional 2-vector space [1]. More precisely, let

$$1 \to k^* \to \widetilde{G} \xrightarrow{\pi} G \to 1$$

be the central extension corresponding to the cocycle c. For every $g \in G$ the set $\pi^{-1}(g)$ is the a k^*-torsor, and therefore

$$L_g := \pi^{-1}(g) \cup \{0\}$$

is a 1-dimensional k-vector space. The group structure on $\widetilde{G}$ induces isomorphisms

$$L_g \otimes_k L_h \to L_{gh}.$$

It follows that associating to $g \in G$ the 2-matrix $\|L_g\|$ of size 1×1 gives a 2-representation of G on $[1] \in \mathrm{Ob}(2\text{-}\mathrm{Vect}_k)$

More generally, a 2-representation ρ of G on $[n]$ consists of the following data: for each $g \in G$, a quasi-invertible 2-matrix $\rho(g) = \|\rho(g)_{ij}\|$ of size $n \times n$, with each $\rho(g)_{ij}$ being a k-vector space, plus the data $\phi_{g,h}$ as in Definition 4.1.

Lemma 5.3. *A 2-matrix $A = \|A_{ij}\|$ of size $n \times n$ is quasi-invertible if and only if there is a permutation $\sigma \in \Sigma_n$ such that $A_{ij} = 0$ for $i \neq \sigma(j)$ and $\dim(A_{i,\sigma(i)}) = 1$.*

It follows that an n-dimensional 2-representation of G defines a homomorphism $G \to \Sigma_n$ plus some cocycle data. This is naturally explained in the context of induced 2-representations, cf. Section 7 below.

Remark 5.4. Lemma 5.3 suggests that the theory becomes richer if one works with generalizations of $2Vect$ that have more interesting quasi-invertible 1-morphisms. One such generalization was defined in [Elgc].

5.3. **Constructible sheaves.** Let X be a real analytic manifold acted upon by G. We view each $g \in G$ as map $g\colon X \to X$. Let $\mathcal{V}$ be the category $\mathcal{D}^b Constr(X)$, see Section 2.4 (d). The base field k will be taken to be the field $\mathbb{C}$ of complex numbers, for simplicity. We have then an action ρ of G on $\mathcal{V}$ given by

$$\rho(g)(\mathcal{F}) = (g^{-1})^*(\mathcal{F})$$

(inverse image under g^{-1}). As in Examples 2.4 (c), (d), it is more practicable to lift this action on a category to an action on an object X of the 2-category $\mathbb{R}\mathcal{A}n_{\mathbb{C}}$. Namely, for $g \in G$ we denote by $\Gamma(g) \subset X \times X$ its graph and associate to g the constructible sheaf $\mathcal{K}_g = \underline{\mathbb{C}}_{\Gamma(g)}$, the constant sheaf on $\Gamma(g)$. Note that $\rho(g) = F_{\mathcal{K}_g}$ is the functor associated to $\mathcal{K}_g$. It is clear that the correspondence $g \mapsto \mathcal{K}_g$ gives an action of G on the object X which we denote $\widetilde{\rho}$.

Proposition 5.5. *Assume that X is oriented. Then the categorical character of $\widetilde{\rho}$ is found as follows:*

$$\mathbb{T}r^\bullet(\widetilde{\rho}(g)) = H^{\bullet + \operatorname{codim} X^g}(X^g, \mathbb{C}),$$

where $X^g \subseteq X$ is the fixed point locus of g.

Proof. As in Example 3.6, we denote by $i : X \to X \times X$ the diagonal embedding and we have

$$\mathbb{T}r(\widetilde{\rho}(g)) = \mathbb{T}r(F_{\mathcal{K}_g}) = \mathbb{H}^\bullet(X, i^!(\mathcal{K}_g))$$

and

$$i^!(\mathcal{K}_g) = \underline{R\Gamma}_\Delta(\mathcal{K}_g) = \underline{\mathbb{C}}_{X^g}[\operatorname{codim}(X^g)].$$

Here $\Delta = i(X)$ is the diagonal in $X \times X$. $\square$

Assume further that X is compact, and G is a Lie group acting smoothly on X. Then Proposition 5.5 can be sheafified as follows. Let

$$Y = \big\{(g, x) \in G \times X \,|\, gx = x\big\}$$

be the "universal" fixed point space. We have the natural embedding and projection

$$G \times X \xleftarrow{\eta} Y \xrightarrow{p} G. \tag{4}$$

Further, the group G acts on the left on $G \times X$, preserving Y, by the formula

$$g_0(g, x) = (g_0 g g_0^{-1}, g_0 x).$$

Thus η is G-equivariant, and so is p, if we consider the action of G on itself by conjugations. Thus the constructible complex of sheaves

$$\mathfrak{T}^\bullet = Rp_* \underline{\mathbb{C}}_Y \tag{5}$$

on G is conjugation equivariant. It can be seen as a sheaf-theoretical version of the categorical trace. Indeed, for any $g \in G$ the complex $\mathfrak{T}^\bullet_g$, the stalk of $\mathfrak{T}^\bullet$ at any $g \in G$ has cohomology

$$H^i(\mathfrak{T}^\bullet_g) = H^i(X^g, \mathbb{C}) = \mathbb{T}r^{\bullet - \mathrm{codim}(X^g)}(\widetilde{\rho}(g)). \tag{6}$$

Example 5.6 (Character sheaves)**.** We specialize to the case when G is a complex semisimple algebraic group and X is the flag variety of G, i.e., the space of all Borel subgroups in G with G-action by conjugation. In this case a class of conjugation equivariant complexes on G was constructed by G. Lusztig in the framework of his theory of character sheaves [L85]. Lusztig's complexes are grouped into "series" labeled by an element w of the Weyl group. We consider here the "principal" series corresponding to $w = 1$. Complexes of this series are defined in terms of the diagram (3) and can be interpreted as categorical traces of certain 2-representations of G, via a twisted version of Proposition 5.5 and the equality (6).

To be precise, recall that all Borel subgroups $B \subset G$ are conjugate and the normalizer of any B is B itself. Therefore, the abelianizations $B/[B, B]$ for different B are canonically identified with each other. Equivalently, we can say that they are all identified with a fixed group T (the "abstract" maximal torus, cf. [CG97], p. 137). Since in our case

$$Y = \{(g, B) \in G \times X : g \in B\},$$

we get a projection $q : Y \to T$ taking (g, B) to the image of g in the abelianization of B. Given a 1-dimensional local system $\mathcal{L}$ on T, the sheaf $q^*\mathcal{L}$ on Y is G-equivariant, and hence the constructible complex

$$\mathfrak{T}^\bullet(\mathcal{L}) = Rp_* q^* \mathcal{L} \tag{7}$$

on G is conjugation invariant. The complex from (5) corresponds to $\mathcal{L} = \underline{\mathbb{C}}_T$. Lusztig's character sheaves (corresponding to $w = 1$) are direct summands of $\mathfrak{T}^\bullet(\mathcal{L})$ in the derived category.

This can be interpreted as follows. Let $\pi : Z \to X$ be the basic affine space of G. It is a principal T-bundle on X isomorphic to the product of the $\mathbb{C}^*$-bundles corresponding to fundamental weights, see, e.g., [BP98], Sect. 2.3.2. The fibers $\pi^{-1}(B), B \in X$, are identified with T uniquely up to a group translation in T. Although the local system $\mathcal{L}$ on T is not T-equivariant (unless it is trivial), each translation takes it into an isomorphic sheaf. Therefore it makes sense to speak about local systems on $\pi^{-1}(B)$ isomorphic to $\mathcal{L}$ on T.

Call an *$\mathcal{L}$-twisted sheaf* on X a sheaf on Z whose restriction on each fiber of π has the form $\mathcal{L}' \otimes V$ where $\mathcal{L}'$ is a local system isomorphic to

$\mathcal{L}$ on T, and V is a $\mathbb{C}$-vector space. Let

$$\mathcal{V}(\mathcal{L}) = D^b_{\mathcal{L}} Constr(X)$$

be the derived category of bounded complexes of $\mathcal{L}$-twisted sheaves on X with constructible cohomology. We have a natural action $\rho_{\mathcal{L}}$ of G on $\mathcal{V}(\mathcal{L})$, and the stalk of $\mathfrak{T}^\bullet(\mathcal{L})$ at g can be related to the categorical trace of $\rho_{\mathcal{L}}(g)$ similarly to (6). As before, to make this precise, we need to lift $\rho_{\mathcal{L}}$ to an action $\widetilde{\rho}_{\mathcal{L}}$ by "kernels" and restrict the kernels to the diagonal. Compare [BP98], Sect. 3.3-4.

6. Representations of finite groupoids

6.1. **Reminder on semisimplicity.** Recall that a groupoid is a category with all morphisms invertible. A groupoid is called finite if it has finitely many objects and morphisms. As before, we fix a field k of characteristic 0 containing all roots of unity. We denote by Vect_k the category of finite-dimensional k-vector spaces.

Definition 6.1. Let G be a finite groupoid. A (finite-dimensional) *representation over* k of G is a functor from G to Vect_k. A morphism between two G-representations is a natural transformation between them. We denote the category of G-representations over k by $\mathcal{R}ep(G)$. Object-wise direct sum and tensor product make $\mathcal{R}ep(G)$ into a bimonoidal category, so that its Grothendieck group $R(G)$ (with respect to the direct sum) is a ring, the *representation ring* of G.

Definition 6.2. Let $\alpha\colon H \to G$ be a map of finite groupoids. Then precomposition with α defines a functor

$$\operatorname{res}|_\alpha\colon \mathcal{R}ep(G) \to \mathcal{R}ep(H).$$

If H is a subgroupoid of G and α is its inclusion we also denote $\operatorname{res}|_\alpha$ by $\operatorname{res}|_H^G$.

Definition 6.3. Let G be a finite groupoid. The groupoid algebra $k[G]$ has as underlying k-vector space the vector space with one basis-element e_g for each morphism g of G. The algebra structure is given by

$$e_g \cdot e_h = \begin{cases} e_{gh} & \text{if } g \text{ and } h \text{ are composable} \\ 0 & \text{else} \end{cases}$$

The categories of G-representations over k and of $k[G]$-modules are then equivalent.

Proposition 6.4. *If $\alpha\colon H \to G$ is an equivalence of groupoids, then*

$$\operatorname{res}|_\alpha\colon \mathcal{R}ep(G) \to \mathcal{R}ep(H)$$

is an equivalence of categories.

Corollary 6.5. *If the groupoids G and H are equivalent, then their groupoid algebras are Morita equivalent.*

Observation 6.6. *Every finite groupoid is equivalent to a disjoint union of groups.*

Proof : Let G be a finite groupoid. Pick a representative for each isomorphism class of objects in G. Consider the inclusion of the disjoint union of the automorphism groups of these representatives in G. By construction, this inclusion is fully faithful and essentially surjective, so it is an equivalence of categories. $\square$

Corollary 6.7. *The groupoid algebra of a finite groupoid G is semi-simple. Thus there is a unique decomposition of representations of G into irreducibles.*

Definition 6.8. (a) Let G be a groupoid. The inertia groupoid $\Lambda(G)$ of G has as objects, the automorphisms of G (i.e., morphisms $u \in \operatorname{Mor}(G)$ whose source and target coincide). For two such morphism u, v there is one morphism in $\Lambda(G)$ from u to v for every morphism g of G with $v = gug^{-1}$.

(b) A class function on a groupoid G is a function defined on isomorphism classes of objects of $\Lambda(G)$.

(c) Let ρ be representation of a finite groupoid G. The character of ρ is the k-valued class function on G given by

$$\chi_\rho([g]) = \operatorname{Trace}(\rho(g)).$$

As before for the case of groups, we denote by $\operatorname{Cl}(G; k)$ the ring of class functions on G.

Corollary 6.9. *Sending a representation to its character is a ring map $R(G) \to \operatorname{Cl}(G; k)$. This map becomes an isomorphism after tensoring with k.*

Proof : The first statement is obvious, the second one follows from Observation 6.6 and Proposition 4.7. $\square$

6.2. **Induced representations of groupoids.**

Definition 6.10. Let $\alpha\colon H \to G$ be a map of groupoids, and let V be a representation of H. Viewing V as a $k[H]$-module, we define the induced G-representation of V by

$$\operatorname{ind}|_\alpha(V) := k[G] \otimes_{k[H]} V.$$

We will sometimes write $\operatorname{ind}|_H^G$ for $\operatorname{ind}|_\alpha$, if the map is obvious.

Note that $\operatorname{ind}|_\alpha$ is left adjoint to $\operatorname{res}|_\alpha$.

Let

$$\alpha\colon H \to G$$

be faithful and essentially surjective, and let ρ be a representation of H. By Observation 6.6, we may assume that G and H are disjoint unions of groups. Then a representation of G can be described one group at a time, so we may as well assume that G is a single group and

$$\alpha\colon H_1 \sqcup \cdots \sqcup H_n \to G$$

is given by a (nonempty) set of injective group maps $\alpha_1, \ldots, \alpha_n$.

In this situation, the induced representation of ρ along α is isomorphic to

$$\operatorname{ind}|_\alpha(\rho_1, \ldots, \rho_n) := \bigoplus \operatorname{ind}|_{\alpha_i}\rho_i.$$

Note also that away from the essential image of α, the induced representation along α is zero.

Proposition 6.11. *Assume that α is a faithful functor, let x be an object of G and $g \in Hom_G(x,x)$. Then the character of the induced representation evaluated at g is given by the formula*

$$\chi_{\text{ind}}(x,g) = \sum_{y \in H_0} \frac{1}{|\operatorname{orbit}_H(y)|} \frac{1}{|\operatorname{Aut}_H(y)|} \sum_{\substack{s \in G_1 | sx=y \\ sgs^{-1} \in \operatorname{Aut}_H(y)}} \chi(y, sgs^{-1}).$$

Here $\operatorname{orbit}_H(y)$ *is the H-isomorphism class of y, and the second sum is over all morphisms s of G with source x and target y that conjugate g into a morphism in the image of $\alpha|_{\operatorname{Aut}_H(y)}$.*

Proof : Let $\operatorname{orbit}_G(x)$ denote the G-isomorphism class of x, and let

$$\mathcal{R} = \{y_1, \ldots, y_n\}$$

be a system of representatives for the H-isomorphism classes mapping to $\operatorname{orbit}_G(x)$ under α. For each j, pick an s_j with $s_j x = \alpha(y_j)$. Denote $\alpha|_{\operatorname{Aut}_H(y_j)}$ by α_j. We have

$$\chi_{\text{ind}} = \sum_{j=1}^n \chi_{\operatorname{ind}|_{\alpha_j}}$$

and

$$\chi_{\mathrm{ind}\,|_{\alpha_j}}(x,g) = \chi_{\mathrm{ind}\,|_{\alpha_j}}(s_j x, s_j g s_j^{-1}).$$

By the classical formula for the character of an induced representation of a group, we have

$$\begin{aligned}
\chi_{\mathrm{ind}|_{\alpha_j}}(s_j x, s_j g s_j^{-1}) &= \frac{1}{|\operatorname{Aut}_H(y_j)|} \sum_{\substack{t\in \operatorname{Aut}_G(y_j)\\ ts_jgs_j^{-1}t^{-1}\in \operatorname{Aut}_H(y_j)}} \chi(y_j, ts_jgs_j^{-1}t^{-1}) \\
&= \frac{1}{|\operatorname{Aut}_H(y_j)|} \sum_{\substack{sx=y_j\\ sgs^{-1}\in \operatorname{Aut}_H(y_j)}} \chi(y_j, sgs^{-1}).
\end{aligned}$$

The first sum in the Proposition is over all objects y of H. If $\alpha(y)$ is not isomorphic to x, then the second sum is empty. For the ones isomorphic to x, we are double counting: rather than just having one summand for the representative y_j, we have the same summand for every element in its H-orbit. □

Corollary 6.12. *Consider an inclusion of groups $H \subseteq G$ and the induced map of inertia groupoids $\alpha\colon \Lambda(H) \to \Lambda(G)$. Let ρ be a representation of $\Lambda(H)$. Then the character of the induced representation* $\mathrm{ind}\,|_{\alpha}\rho$ *evaluated at a pair of commuting elements of G is given by the formula*

$$\begin{aligned}
\chi(g_1, g_2) &= \sum_{h_1\in H} \frac{1}{|[h_1]_H|}\frac{1}{|C_H(h_1)|} \sum_{\substack{sg_1s^{-1}=h_1\\ sg_2s^{-1}\in C_H(h_1)}} \chi(sg_1s^{-1}, sg_2s^{-1}) \\
&= \frac{1}{|H|} \sum_{\substack{s(g_1,g_2)s^{-1}\\ \in H\times H}} \chi(sg_1s^{-1}, sg_2s^{-1}).
\end{aligned}$$

Here $[h_1]_H$ is the conjugacy class of h_1 in H while $C_H(h_1)$ is the centralizer of h_1 in H.

7. Induced 2-representations

7.1. Main definitions.

Definition 7.1. Let $H \subseteq G$ be an inclusion of finite groups. Let

$$\rho\colon H \to \mathrm{Fun}(\mathcal{V}, \mathcal{V})$$

be an action of H on a category $\mathcal{V}$. Let $\mathrm{ind}\,|_H^G(\mathcal{V})$ be the category whose objects are maps

$$f\colon G \to \mathrm{Ob}\mathcal{V}$$

together with an isomorphism for every $g \in G$ and $h \in H$

$$u_{g,h} \colon f(gh) \to \rho(h^{-1})(f(g))$$

satisfying the following two conditions: First, it is required that

$$u_{g,1} : f(g) \to \rho(1)(f(g))$$

coincides with $\phi^{-1}_{1,f(g)}$, see Definition 4.1(d). Second, it is required that for every $g \in G$ and every $h_1, h_2 \in H$, the diagram

$$\begin{tikzcd}
f(gh_1h_2) \arrow[r, "u_{gh_1,h_2}"] \arrow[d, "u_{g,h_1h_2}"'] & \rho(h_1^{-1})(f(gh_2)) \arrow[d, "\rho(h_2^{-1})u_{g,h_1}"] \\
\rho((h_1h_2)^{-1})(f(g)) & \rho(h_2^{-1})\rho(h_1^{-1})(f(g)) \arrow[l, "\phi_{h_1^{-1},h_2^{-1}}"]
\end{tikzcd}$$

commutes.

A morphism in $\mathrm{ind}\,|^G_H(\mathcal{V})$ between two systems $(f, u = (u_{g,h}))$ and $(f', u' = (u'_{g,h}))$ is a system of morphisms $f(g) \to f'(g)$ in $\mathcal{V}$, given got each $g \in G$ and commuting with the $u_{g,h}$ and $u'_{g,h}$.

We define a left action σ of G on $\mathrm{ind}\,|^G_H(\mathcal{V})$ by

$$(\sigma(g_1)f)(g) = f(g_1^{-1}g), \quad (\sigma(g_1)u)_{g,h} = u_{g_1^{-1}g,h}.$$

Remark 7.2. Consider the category $\prod_{g\in G}\mathcal{V}$ whose objects are all maps $G \to \mathrm{Ob}\mathcal{V}$. This category has a left H-action ξ given by

$$(\xi(h)f)(g) = \rho(h)(f(gh)).$$

One sees immediately that

$$\mathrm{ind}\,|^G_H(\mathcal{V}) = \big(\prod_{g\in G}\mathcal{V}\big)^H$$

is identified with the category of H-equivariant objects in $\prod_{g\in G}\mathcal{V}$.

An explicit description of $\mathrm{ind}\,|^G_H\mathcal{V}$ is given as follows (compare this to the classical definition of the induced representation as in, e.g., [Ser77]):

Let m be the index of H in G. The underlying category of $\mathrm{ind}\,|^G_H\mathcal{V}$ is then identified with $\mathcal{V}^m$. Such an identification is obtained by picking a system of representatives

$$\mathcal{R} = \{r_1, \dots r_m\}$$

of left cosets of H in G and associating to every map f as above the system $(f(r_1), ..., f(r_m))$.

We view $\mathrm{ind}\,|_H^G \rho(g)$ as $m \times m$ matrix whose entries are functors from $\mathcal{V}$ to $\mathcal{V}$. Then

$$\left(\mathrm{ind}\,|_H^G \rho(g)\right)_{ij} = \begin{cases} \rho(h) & \text{if} \quad gr_j = r_i h, h \in H \\ 0 & \text{else.} \end{cases}$$

Note that in each row and each column, there is exactly one block entry, and that therefore such a matrix gives a functor from $\mathcal{V}^m$ to $\mathcal{V}^m$.
Composition is defined as follows:

$$\left(\mathrm{ind}\,|_H^G \rho(g_1)\right) \circ_1 \left(\mathrm{ind}\,|_H^G \rho(g_2)\right)_{ik} =$$

$$= \begin{cases} \rho(h_1) \circ_1 \rho(h_2) & \text{if} \quad g_1 r_j = r_i h_1 \text{ and } g_2 r_k = r_j h_2 \\ 0 & \text{else.} \end{cases}$$

In the case that this is not zero,

$$\left(\mathrm{ind}\,|_H^G \rho(g_1 g_2)\right)_{ik} = \rho(h_1 h_2),$$

since in this case

$$g_1 g_2 r_k = r_i h_1 h_2.$$

On this block the composition isomorphism is given by the 2-isomorphism

$$\rho(h_1) \circ_1 \rho(h_2) \Rightarrow \rho(h_1 h_2).$$

Similarly, the isomorphism

$$\mathrm{ind}\,|_H^G \rho(1) \Rightarrow 1_{\mathcal{V}^m}$$

is given by the corresponding map for ρ.

Proposition 7.3 (compare [Ost01, Ex.3.4.,Th.2] and [Elgb, 6.5])**.** *Let G be a finite group, and let $\mathcal{V} = \mathrm{Vect}_k^{\oplus n}$. Then any 2-representation ρ of G in $\mathcal{V}$ is isomorphic to a direct sum*

$$\bigoplus_{i=1}^m \mathrm{ind}\,|_{H_i}^G \rho_{\omega_i},$$

where the H_i are subgroups of G, $\omega_i \in H^2(H_i, k^*)$, and ρ_{ω_i} is the 1-dimensional 2-representation corresponding to ω_i. Moreover, the system of (H_i, ω_i) is determined by the G-action on $\mathcal{V}$ uniquely up to conjugation.

Proof : By Lemma 5.3, ρ defines a homomorphism from G to Σ_n, i.e., a G-action on the set $\{1, \dots, n\}$. Let $O_1, \dots, O_m$ be the orbits of this action. It follows that, after renumbering of $1, \dots, n$, the 2-matrices $\rho(g)$ become block diagonal with blocks of sizes $|O_1|, \dots, |O_m|$. Hence

$$\rho \cong \bigoplus_{i=1}^m \rho_i.$$

Let H_i be the stabilizer of an element of O_i. For $h \in H_i$, the 2-matrix $\rho_i(h)$ is diagonal with the same 1-dimensional vector space $L_i(h)$ on the diagonal. We conclude that

$$\rho_i \cong \operatorname{ind}|_{H_i}^G(\rho_{\omega_i}),$$

where ρ_{ω_i} is the 1-dimensional 2-representation of H_i corresponding to the system $(L_i(h), h \in H_i)$. □

Remark 7.4. It follows that in the particular case of the proposition, the approach of Ostrik is equivalent to ours. In fact, the category

$$\left(\operatorname{ind}|_H^G(\rho_\omega)\right)^G, \quad H \subseteq G, \quad \omega \in H^2(H, k^*),$$

is identified, as a $\mathcal{R}ep(G)$-module category, with the category of projective representations of H with the central charge ω.

7.2. **The character of the induced 2-representation.** The aim of this subsection is to prove the following theorem.

Theorem 7.5. *Let $\mathcal{V}$ be k-linear. The categorical trace $\mathbb{T}r$ takes induced 2-representations into induced representations of groupoids. That is,*

$$\mathbb{T}r(\operatorname{ind}|_H^G \rho) \cong \operatorname{ind}|_{\Lambda(H)}^{\Lambda(G)}(\mathbb{T}r(\rho)),$$

as representations of $\Lambda(G)$.

Corollary 7.6 (compare [HKR00, Thm D])**.** *Assume that the $\mathbb{T}r(\rho(h))$ are finite dimensional. Let χ denote the 2-character of ρ. The 2-character of the induced representation is given by*

$$\chi_{\text{ind}}(g, h) = \frac{1}{|H|} \sum_{s^{-1}(g,h)s \in H \times H} \chi(s^{-1}gs, s^{-1}hs).$$

Proof of Theorem 7.5: We want to compute

$$\chi_{\text{ind}} := \mathbb{T}r(\operatorname{ind}|_H^G \rho)$$

as representation of

$$\Lambda(G) \simeq \coprod_{[g]_G} C_G(g).$$

Let $\mathcal{R}$ be a system of representatives of G/H, and fix $g \in G$. The underlying vector space of $\chi_{\text{ind}}(g)$ is the sum over all $r \in \mathcal{R}$ which produce a diagonal block entry in $\operatorname{ind}_\rho(g)$,

$$\chi_{\text{ind}}(g) = \bigoplus_{r^{-1}gr \in H} \mathbb{T}r(\rho(r^{-1}gr)), \tag{8}$$

(compare [Ser77]). We need to determine the action of $C_G(g)$ on $\chi_{\text{ind}}(g)$. For this purpose, we replace our system of representatives $\mathcal{R}$ in a convenient way: The decomposition

$$[g]_G \cap H = [h_1]_H \cup \cdots \cup [h_l]_H$$

induces a decomposition

$$\{r \in \mathcal{R} \mid r^{-1}gr \in H\} = \bigcup_{i=1}^{l} \mathcal{R}_i,$$

with

$$\mathcal{R}_i := \{r \in \mathcal{R} \mid r^{-1}gr \in [h_i]_H\}.$$

We fix i, pick $r_i \in \mathcal{R}_i$, and write $h_i := {r_i}^{-1}gr_i$.

Lemma 7.7. *We can replace the elements of $\mathcal{R}_i$ in such a way that left multiplication with r_i^{-1} maps $\mathcal{R}_i$ bijectively into a system of representatives of*

$$C_G(h_i)/C_H(h_i).$$

Proof : If $r \in \mathcal{R}_i$ satisfies

$$r^{-1}gr = h^{-1}h_ih,$$

we replace r by rh^{-1}, which represents the same left coset of G/H as r does. Note that

$$(rh^{-1})^{-1}grh^{-1} = h_i. \tag{9}$$

We have

$$(r_i^{-1}rh^{-1})^{-1}h_i(r_i^{-1}rh^{-1}) = h_i,$$

therefore $r_i^{-1}(rh^{-1}) \in C_G(h_i)$. Assume now that we have replaced $\mathcal{R}_i$ in this way. To prove that left multiplication with r_i^{-1} is injective, let $r \neq r' \in \mathcal{R}_i$. Then

$$(r_i^{-1}r)^{-1}r_i^{-1}r' = r^{-1}r'$$

is not in H, and therefore $r_i^{-1}r'$ and $r_i^{-1}r$ are in different left cosets of $C_H(h_i)$ in $C_G(h_i)$. To prove surjectivity, let $\tilde{g} \in C_G(h_i)$. Write

$$r_i\tilde{g} = rh$$

with $r \in \mathcal{R}$ and $h \in H$. Then

$$r^{-1}gr = h\tilde{g}^{-1}{r_i}^{-1}gr_i\tilde{g}h^{-1} = h\tilde{g}^{-1}h_i\tilde{g}h^{-1} = hh_ih^{-1}.$$

Therefore, r is in $\mathcal{R}_i$, and it follows from the identity (9) that

$$r^{-1}gr = h_i.$$

Thus $r_i^{-1}r = \tilde{g}h$ is in the same left coset of $C_H(h_i)$ in $C_G(h_i)$ as $\tilde{g}$ is. $\square$

Let α_i denote the composition

$$C_H(h_i) \hookrightarrow C_G(h_i) \to C_G(g),$$

where the second map is conjugation by r_i^{-1}. Recall that as representation of $C_G(g)$,

$$\left(\operatorname{ind}|_{\Lambda(H)}^{\Lambda(G)}\pi\right)(g) \quad = \quad \bigoplus_{i=1}^{l} \operatorname{ind}|_{\alpha_i}\pi(h_i). \tag{10}$$

Lemma 7.8. *As a representation of C_G,*

$$\chi_{\text{ind}}(g) \cong \bigoplus_{i=1}^{l} \operatorname{ind}|_{\alpha_i}\mathbb{T}r(\rho(h_i)).$$

Proof : Let $f \in C_G(g)$, and let $r \in \mathcal{R}_i$. Write

$$fr = \tilde{r}h,$$

with $\tilde{r} \in \mathcal{R}$ and $h \in H$. We claim that $\tilde{r}$ is also in $\mathcal{R}_i$ and that h is in $C_H(h_i)$. This follows from

$$\tilde{r}^{-1}g\tilde{r} = hr^{-1}f^{-1}gfrh^{-1} = hr^{-1}grh^{-1} = hh_ih^{-1} = h_i$$

as in the proof of Lemma 7.7. We are now ready to compute the block entry corresponding to (r, r) of

$$\operatorname{ind}_\rho(f^{-1}) \circ_1 \operatorname{ind}_\rho(g) \circ_1 \operatorname{ind}_\rho(f) :$$

$$\begin{aligned} fr = \tilde{r}h \quad &\text{gives} \quad (\operatorname{ind}_\rho(f))_{\tilde{r}r} = \rho(h), \\ g\tilde{r} = \tilde{r}h_i \quad &\text{gives} \quad (\operatorname{ind}_\rho(g))_{\tilde{r}\tilde{r}} = \rho(h_i), \\ f^{-1}\tilde{r} = rh^{-1} \quad &\text{gives} \quad (\operatorname{ind}_\rho(f^{-1}))_{r\tilde{r}} = \rho(h^{-1}) \end{aligned}$$

and all other block entries in these rows and columns are zero. Thus

$$\left(\operatorname{ind}_\rho(f^{-1}) \circ_1 \operatorname{ind}_\rho(g) \circ_1 \operatorname{ind}_\rho(f)\right)_{rr} = \rho(h^{-1}) \circ_1 \rho(h_i) \circ_1 \rho(h), \tag{11}$$

and the 2-morphism from (11) to

$$\left(\operatorname{ind}_\rho(g)\right)_{rr} = \rho(h_i)$$

is the 2-morphism

$$\rho(h^{-1}) \circ_1 \rho(h_i) \circ_1 \rho(h) \Rightarrow \rho(h_i)$$

corresponding to $h^{-1}h_ih = h_i$. This proves that the action of $C_G(g)$ on $\chi_{\text{ind}}(g)$ decomposes into actions on

$$\bigoplus_{r \in \mathcal{R}_i} \rho(h_i).$$

More precisely, if $fr = \tilde{r}h$, then f maps the summand corresponding to r to the one corresponding to $\tilde{r}$ by

$$h\colon \rho(h_i) \to \rho(h_i).$$

But

$$fr = \tilde{r}h \iff ({r_i}^{-1}fr_i)(r_i^{-1}r) = (r_i^{-1}\tilde{r})h,$$

and the action of

$${r_i}^{-1}fr_i \in C_G(h_i)$$

on

$$\operatorname{ind}|_{C_H(h_i)}^{C_G(h_i)}\rho(h_i)$$

is given by

$$h\colon r_i^{-1}r\rho(h_i) \to r_i^{-1}\tilde{r}\rho(h_i).$$

Lemma 7.8 is proved. □

This now completes the proof of Theorem 7.5. □

8. Some further questions

8.1. **Inertia orbifolds.** Recall that a *Lie groupoid* is a groupoid Γ enriched in the category of C^∞-manifolds, i.e., such that $\mathrm{Ob}(\Gamma)$ and $\mathrm{Mor}(\Gamma)$ are C^∞-manifolds and all the structure maps (composition, inverses, units) are smooth. See [Mack05] for more details. An *orbifold*, cf. [Moe02], is a Lie groupoid G such that all stabilizer groups

$$Hom_\Gamma(x, x), \quad x \in \mathrm{Ob}(\Gamma),$$

are finite. The construction of an inertia groupoid $\Lambda(\Gamma)$, see Definition 6.8, can be applied to a Lie groupoid (resp. orbifold) Γ and the result is again a Lie groupoid (resp. orbifold).

Example 8.1 (Global quotient groupoids)**.** Let M be a manifold and G be a Lie group acting on M. Then we have a Lie groupoid $M/\!\!/G$ with

$$\mathrm{Ob}(M/\!\!/G) = M, \quad Hom_{M/\!\!/G}(x, y) = \{g \in G : g(x) = y\}.$$

Thus $\mathrm{Mor}(M/\!\!/G) = G \times M$. If the stabilizer of each $x \in M$ is finite, then $M/\!\!/G$ is an orbifold, known as the *global quotient orbifold*. The latter condition is automatically satisfied, if G itself is finite. In this case the inertia orbifold of $M/\!\!/G$ can be identified as follows:

$$\Lambda(M/\!\!/G) = \coprod_{[g]_G} M^g/\!\!/C_G(g).$$

Here the disjoint union is over the conjugacy classes in G, and M^g stands for the g-fixed point locus of M.

Recall further that equivariant K-theory of a manifold with a finite group action is a particular case of a more general concept of orbifold K-theory $K_{\mathrm{orb}}(\Gamma)$ defined for any orbifold Γ. This particular case corresponds to a global quotient orbifold:

$$K_{\mathrm{orb}}(M/\!\!/G) \cong K_G(M).$$

Our 2-character map

$$\mathbb{T}r\colon 2\,\mathrm{Rep}(G) \to \mathrm{Rep}(\Lambda(G)) \tag{12}$$

should be compared to the orbifold Chern character map defined by Adem-Ruan and interpreted by Moerdijk [Moe02, p. 18] as a map

$$K^{\bullet}(\Gamma) \otimes \mathbb{C} \to \prod_i H^{2i+\bullet}(\Lambda(\Gamma), \mathbb{C}). \tag{13}$$

Here Γ is any orbifold whose quotient space (i.e., the space of isomorphism classes of objects) is compact.

This suggests that (12) has a generalization for an arbitrary orbifold Γ as above, yielding a transformation

$$2K_{\mathrm{orb}}(\Gamma) \to K_{\mathrm{orb}}(\Lambda(\Gamma)).$$

Here $2K_{\mathrm{orb}}(\Gamma)$ is a (yet to be defined) orbifold/equivariant version of the 2-vector bundle K-theory of [BDR04]. Recall that the non-orbifold $2K$ is interpreted as some approximation to elliptic cohomology. Therefore the orbifold version is to be regarded as a geometric version of equivariant elliptic cohomology, thus making more precise our point in the introduction. Note that inertia orbifolds also turn up in the original paper [HKR00], where working at chromatic level n requires using n-fold iterated inertia orbifolds $\Lambda^n(\Gamma)$.

8.2. **The Todd genus of X^g.** Let G be a finite group acting on a compact d-dimensional complex manifold X. For $g \in G$, the fixed point locus X^g is then a compact complex submanifold. Consider the graded vector spaces

$$\mathbb{T}(g) := H^{\bullet}(X^g, \mathcal{O}),$$

where $\mathcal{O}$ is the sheaf of holomorphic functions on X^g. Clearly, these are conjugation equivariant, i.e., they form a representation of $\Lambda(G)$. Its character is a 2-class function

$$\chi_X \in 2\,\mathrm{Cl}(G; \mathbb{Z}[\zeta_N]), \quad \chi_X(g,h) = \mathrm{Trace}\big\{h_* : \mathbb{T}(g) \to \mathbb{T}(g)\big\}.$$

This function takes values in the cyclotomic ring $\mathbb{Z}[\zeta_N]$, where ζ_N is an N^{th} root of 1 and N is the order of G.

On the other hand, the Hopkins-Kuhn-Ravenel theory [HKR00] also provides a 2-class function associated with the G-manifold X. Let

$[X] \in \mathrm{MU}(\mathrm{BG})$ denote the image of the equivariant cobordism class of X. Fix a prime p, and let $E = E_2$ be the second Morava E-theory at p. Recall that E comes with a canonical natural transformation of cohomology theories

$$\phi\colon \mathrm{MU}^*(-) \to E^*(-).$$

Let now G be finite. In this situation Hopkins, Kuhn and Ravenel constructed a map

$$\alpha : E^*(BG) \to 2\,\mathrm{Cl}(G; D),$$

for a certain ring

$$D = \varinjlim_n D_n,$$

where D_n is known as the ring of Drinfeld level p^n structures on the formal group $E^*(\mathrm{pt})$:

$$D_n := E^0(B(\mathbb{Z}/p^n\mathbb{Z})^2)/(\text{annihilators of nontrivial Euler classes}).$$

The ring D_n can be seen as the second chromatic analog of the cyclotomic ring $\mathbb{Z}[\zeta_{p^n}]$ which corresponds to level p^n structures on the multiplicative group. In fact, a version of the Weil pairing [AS01] shows that D_n contains $\mathbb{Z}[\zeta_{p^n}]$. The 2-class function $a(y)$, $y \in E^*(BG)$ is defined by

$$a(y)(g,h) := (g,h)_n^*(y) \in D_n$$

where (g,h) is a pair of commuting p-power order elements of G and

$$(g,h)_n : (\mathbb{Z}/p^n\mathbb{Z})^2 \to G, \quad p^n = \max(\mathrm{ord}(g), \mathrm{ord}(h)),$$

is the homomorphism corresponding to (g,h).

Question 1: Is there a natural 2-representation in some category of sheaves associated to X whose categorical character is $\mathbb{T}$?

Question 2: What is the relationship between $a(\phi[X])(g,h)$ and $\chi_X(g,h)$?

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